

GENERIC UNIPOTENT STANDARD MODULES

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ABSTRACT. Using Lusztig's geometric classification, we find the reducibility points of a standard module for the affine Hecke algebra, in the case when the inducing data is generic. This recovers the known result of [MS] for representations of split p -adic groups with Iwahori-spherical Whittaker vectors. We also give a necessary (insufficient) condition for reducibility in the non-generic case.

By [L6], the unipotent representations of a split p -adic group \mathcal{G} of adjoint type are classified in terms of geometric data for the dual complex group G . More precisely, they are indexed by certain triples $(\chi, \mathcal{O}, \mathcal{L})$, where χ is a Weyl orbit of semisimple elements in G , \mathcal{O} is a “graded” orbit in the Lie algebra \mathfrak{g} , and \mathcal{L} is a local system on \mathcal{O} . This is realized via equivalences with module categories for affine Hecke algebras of *geometric type* constructed from G ([L2, L3]). It is shown in [R], that in this correspondence, the unipotent representations of \mathcal{G} admitting Whittaker vectors (*generic*) correspond to maximal orbits \mathcal{O} and trivial \mathcal{L} . For Iwahori-spherical representations, the same result, with a different proof, follows from [BM1] (and [BM2]).

In this paper, we determine explicitly, as a consequence of the geometric classification, the reducibility points for the standard representations (in the sense of Langlands classification) when the inducing data is generic. This was known from [MS], as a consequence of the Langlands-Shahidi method. In particular, our main result, theorem 3.2 is essentially the same as proposition 3.3 in [MS] (our parameter ν corresponds to the parameter s in there). We also show that for non-generic inducing data, the reducibility points are necessarily a subset of those for the corresponding generic case.

For simplicity, we will work in the setting of the affine *graded Hecke algebra* of [L1], and real central character (section 1.2), from which one can recover the representation theory of the affine Hecke algebra (see section 4 in [L6] for example). Most of the paper is devoted to recording the relevant geometric results, particularly from [L7]. Once they are in place, the reducibility follows immediately by a simple comparison of dimensions of orbits.

The information about reducibility of standard modules played an important role in the determination of the generic Iwahori-spherical unitary dual (equivalently, spherical unitary dual) of split p -adic groups of exceptional types in [BC]. In fact, this paper is mainly motivated by that work.

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1. GRADED HECKE ALGEBRA

1.1. Let \mathfrak{h} be a finite dimensional vector space, $R \subset \mathfrak{h}^*$ a root system, with $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots, $\check{R} \subset \mathfrak{h}$ the set of coroots, and W the Weyl group. Let $c : R \rightarrow \mathbb{Z}_{>0}$ be a function such that $c_\alpha = c_\beta$, whenever α and β are W -conjugate. As a vector space,

$$\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{A}, \quad (1.1.1)$$

where \mathbb{A} is the symmetric algebra over \mathfrak{h}^* . The generators are $t_w \in \mathbb{C}[W]$, $w \in W$ and $\omega \in \mathfrak{h}^*$. The relations between the generators are:

$$\begin{aligned} t_w t_{w'} &= t_{ww'}, & \text{for all } w, w' \in W; \\ t_s^2 &= 1, & \text{for any simple reflection } s \in W; \\ \omega t_s &= t_s s(\omega) + c_\alpha \omega(\check{\alpha}), & \text{for simple reflections } s = s_\alpha. \end{aligned} \quad (1.1.2)$$

1.2. By [L1], the center of \mathbb{H} is \mathbb{A}^W . On any simple (finite dimensional) \mathbb{H} -module, the center of \mathbb{H} acts by a character, which we will call a *central character*. The central characters correspond to W -conjugacy classes of semisimple elements $\chi \in \mathfrak{h}$. We will assume throughout the paper that the characters are *real*, i.e., hyperbolic.

1.3. We present the *Langlands classification* for \mathbb{H} as in [E]. If V is a (finite dimensional) simple \mathbb{H} -module, \mathbb{A} induces a generalized weight space decomposition $V = \bigoplus_{\lambda \in \mathfrak{h}} V_\lambda$. Call λ a *weight* of V if $V_\lambda \neq 0$.

Definition. The irreducible module σ is called *tempered* if $\omega_i(\lambda) \leq 0$, for every weight $\lambda \in \mathfrak{h}$ of σ and every fundamental weight $\omega_i \in \mathfrak{h}^*$, and in addition, λ is zero on the real span of the set $x \in \mathfrak{h}^*$ perpendicular on coroots. If σ is tempered, and $\omega_i(\lambda) < 0$, for all λ, ω_i as above, σ is called a *discrete series*.

For every $\Pi_P \subset \Pi$, define $R_M \subset R$ to be the set of roots generated by Π_P , $\check{R}_M \subset \check{R}$ the corresponding set of coroots, and $W_P \subset W$ the corresponding Weyl subgroup. (The notation will make more sense in the sequel, when $P = MN$ will denote a parabolic subgroup of the complex reductive group G .)

Let \mathbb{H}_M be the Hecke algebra attached to (\mathfrak{h}, R_M) . It can be regarded naturally as a subalgebra of \mathbb{H} .

Define $\mathfrak{t} = \{\nu \in \mathfrak{h} : \langle \alpha, \nu \rangle = 0, \text{ for all } \alpha \in \Pi_P\}$ and $\mathfrak{t}^* = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \check{\alpha} \rangle = 0, \text{ for all } \alpha \in \Pi_P\}$. Then \mathbb{H}_M decomposes as

$$\mathbb{H}_M = \mathbb{H}_{M_s} \otimes S(\mathfrak{t}^*),$$

where \mathbb{H}_{M_s} is the Hecke algebra attached to $(\mathbb{C}\langle \Pi_P \rangle, R_M)$.

We will denote by $I(P, U)$ the induced module $I(P, U) = \mathbb{H} \otimes_{\mathbb{H}_M} U$.

Theorem ([E]). (1) Every irreducible \mathbb{H} -module is a quotient of a standard induced module $X(P, \sigma, \nu) = I(P, \sigma \otimes \mathbb{C}_\nu)$, where σ is a tempered module for \mathbb{H}_{M_s} , and $\nu \in \mathfrak{t}^+ = \{\nu \in \mathfrak{t} : \alpha(\nu) > 0, \text{ for all } \alpha \in \Pi \setminus \Pi_P\}$.
 (2) Assume the notation from (1). Then $X(P, \sigma, \nu)$ has a unique irreducible quotient, denoted by $L(P, \sigma, \nu)$.
 (3) If $L(P, \sigma, \nu) \cong L(P', \sigma', \nu')$, then $\Pi_P = \Pi_{P'}$, $\sigma \cong \sigma'$ as \mathbb{H}_{M_s} -modules, and $\nu = \nu'$.

We will call a triple (P, σ, ν) as in theorem 1.3, a *Langlands parameter*.

2. GEOMETRIC PARAMETERIZATION

Notation. In the following, whenever Q denotes a complex Lie group, Q^0 will be the identity component, and \mathfrak{q} will denote the Lie algebra. If s is an element of Q or \mathfrak{q} , we will denote by $Z_Q(s)$ the centralizer in Q of s .

2.1. Let G be a reductive connected complex algebraic group, with Lie algebra \mathfrak{g} . Let B be a Borel subgroup, and $A \subset B$ a maximal torus. Let $S = LU$ denote a parabolic subgroup, with $\mathfrak{s} = \mathfrak{l} + \mathfrak{u}$ the corresponding Lie algebras, such that \mathfrak{l} admits an irreducible L -equivariant cuspidal local system (as in [L2],[L5]) Ξ on a nilpotent L -orbit $\mathcal{C} \subset \mathfrak{l}$. The classification of cuspidal local systems can be found in [L5]. In particular, $W = N(L)/L$ is a Coxeter group.

Let H be the center of L with Lie algebra \mathfrak{h} , and let R be the set of nonzero weights α for the ad -action of \mathfrak{h} on \mathfrak{g} , and $R^+ \subset R$ the set of weights for which the corresponding weight space $\mathfrak{g}_\alpha \subset \mathfrak{u}$. For each parabolic $S_j = L_j U_j$, $j = 1, n$, such that $S \subset S_j$ maximally and $L \subset L_j$, let $R_j^+ = \{\alpha \in R^+ : \alpha(\mathfrak{z}(\mathfrak{l}_j)) = 0\}$, where $\mathfrak{z}(\mathfrak{l}_j)$ denotes the center of \mathfrak{l}_j . It is shown in [L2] that each R_j^+ contains a unique α_j such that $\alpha_j \notin 2R$.

Let $Z_G(\mathcal{C})$ denote the centralizer in G of a Lie triple for \mathcal{C} , and $\mathfrak{z}(\mathcal{C})$ its Lie algebra.

Proposition ([L2]). (1) R is a (possibly non-reduced) root system in \mathfrak{h}^* , with simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, with Weyl group W .
 (2) H is a maximal torus in $Z^0 = Z_G^0(\mathcal{C})$.
 (3) W is isomorphic to $W(Z_G^0(\mathcal{C})) = N_{Z^0}(H)/H$.
 (4) The set of roots in $\mathfrak{z}(\mathcal{C})$ with respect to \mathfrak{h} is exactly the set of non-multipliable roots in R .

For each $j = 1, \dots, n$, let $d_j \geq 2$ be such that

$$(ad(e)^{d_j-2} : \mathfrak{l}_j \cap \mathfrak{u} \rightarrow \mathfrak{l}_j \cap \mathfrak{u}) \neq 0, \text{ and } (ad(e)^{d_j-1} : \mathfrak{l}_j \cap \mathfrak{u} \rightarrow \mathfrak{l}_j \cap \mathfrak{u}) = 0. \quad (2.1.1)$$

By proposition 2.12 in [L2], $d_i = d_j$ whenever α_i and α_j are W -conjugate. Therefore, as in (1.1.1),(1.1.2), we can define a Hecke algebra \mathbb{H}_S with parameters $c_j = d_j/2$. The explicit algebras which may appear are listed in 2.13 of [L2]. (The case of Hecke algebras with equal parameters arises when one takes $S = B$, and \mathcal{C} and Ξ to be trivial.)

If $P \subset G$ is a parabolic subgroup, such that $S \subset P$, then denote

$$\Pi_{P/S} = \{\alpha_j \in \Pi : S_j \subset P\}. \quad (2.1.2)$$

When $S = B$, we write just Π_P .

Let us denote by $\Phi(G)$ the set of graded Hecke algebras \mathbb{H}_S obtained by the above construction. The unique Hecke algebra with equal parameters in $\Phi(G)$ will be denoted by \mathbb{H}_0 .

2.2. Let \mathfrak{a} be the Lie algebra of the maximal torus $A \subset G$. Denote by Δ the roots of A in G , and let \mathfrak{g}_α be the root subspace corresponding to $\alpha \in \Delta$.

Fix a (hyperbolic) semisimple element (an infinitesimal character) $\chi \in \mathfrak{a}$, and set

$$G_0 = \{g \in G : Ad(g)\chi = \chi\}, \quad \mathfrak{g}_t = \{y \in \mathfrak{g} : [\chi, y] = ty\}, \quad t \in \mathbb{R}. \quad (2.2.1)$$

Note that

$$\mathfrak{g}_t = \begin{cases} \bigoplus_{\alpha \in \Delta, \alpha(\chi)=t} \mathfrak{g}_\alpha, & t \neq 0 \\ \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta, \alpha(\chi)=0} \mathfrak{g}_\alpha, & t = 0 \end{cases}. \quad (2.2.2)$$

For $\mathbb{H} \in \Phi(G)$, corresponding to $S = LU$ in the notation of section 2.1, denote by $\text{mod}_\chi \mathbb{H}$ the category of finite dimensional \mathbb{H} -modules of central character congruent to χ modulo $\mathfrak{z}(\mathfrak{l})$.

Theorem ([L3]). *There exists a one-to-one correspondence between the standard (or irreducible) objects in $\sqcup_{\mathbb{H} \in \Phi(G)} \text{mod}_\chi(\mathbb{H})$ and the set of pairs $\xi = (\mathcal{O}, \mathcal{L})$, where*

- (1) \mathcal{O} is a G_0 -orbit on \mathfrak{g}_1 .
- (2) \mathcal{L} is an irreducible G_0 -equivariant local system on \mathcal{O} .

We say that two modules in $\sqcup_{\mathbb{H} \in \Phi(G)} \text{mod}_\chi(\mathbb{H})$ are in the same L -packet if they correspond to the same orbit \mathcal{O} .

For the \mathbb{H}_0 -modules, the local systems which appear are of *Springer type* ([L7]). More precisely, if $e \in \mathcal{O}$, then \mathcal{L} corresponds to a representation ϕ of the component group $Z_{G_0}(e)/Z_{G_0}(e)^0$. The representations ϕ which are allowed must be in the restriction $Z_{G_0}(e)/Z_{G_0}(e)^0 \subset Z_G(e)/Z_G(e)^0$ of a representation which appears in Springer's correspondence.

2.3. Let $\text{Orb}_1(\chi)$ denote the set of G_0 orbits on \mathfrak{g}_1 .

- (1) $\text{Orb}_1(\chi)$ is finite.
- (2) For every $\mathcal{O} \in \text{Orb}_1(\chi)$, $\overline{\mathcal{O}} \setminus \mathcal{O}$ is the union of some orbits \mathcal{O}' with $\dim \mathcal{O}' < \dim \mathcal{O}$.
- (3) There is a unique open (dense) orbit $\mathcal{O}_{\text{open}}$ in $\text{Orb}_1(\chi)$.

In other words, \mathfrak{g}_1 is a prehomogeneous vector space with finitely many G_0 -orbits. A parameterization for $\text{Orb}_1(\chi)$ appeared in [K]. We will instead use, in sections 2.6 and 2.7, the formulation of [L7].

2.4. By [L4], the categories $\text{mod}_\chi \mathbb{H}$, $\mathbb{H} \in \Phi(G)$, have tempered modules if and only if χ is the middle element of a nilpotent orbit in \mathfrak{g} . In this case the standard modules parameterized by $(\mathcal{O}_{\text{open}}, \mathcal{L})$ are irreducible and they exhaust the tempered modules. If in addition, χ is the middle element of a distinguished nilpotent orbit, then the tempered modules are discrete series.

2.5. By [R], there is a unique *generic* module in $\sqcup_{\mathbb{H} \in \Phi(G)} \text{mod}_\chi(\mathbb{H})$, which is parametrized by $(\mathcal{O}_{\text{open}}, \text{triv})$, where *triv* denotes the trivial local system. Note that this is always a module of \mathbb{H}_0 . The fact that the generic module in $\text{mod}_\chi(\mathbb{H}_0)$ is parameterized by $(\mathcal{O}_{\text{open}}, \text{triv})$ is also an immediate consequence of the results in [BM1] and [BM2]. In [BM1], it is proven that the generic \mathbb{H}_0 -module is characterized by the property that it contains the *sign* representation of W .

2.6. Let e be a representative of an orbit $\mathcal{O} = \mathcal{O}_e$ in \mathfrak{g}_1 . To e , one associates, conform [L7], a parabolic subalgebras of \mathfrak{g} , which we'll denote \mathfrak{p}^e . This is used to give a parameterization of $\text{Orb}_1(\chi)$.

By the graded version of the Jacobson-Morozov triple ([L7]), $e \in \mathfrak{g}_1$ can be embedded into a Lie triple $\{e, h, f\}$, such that $h \in \mathfrak{a} \subset \mathfrak{g}_0$, and $f \in \mathfrak{g}_{-1}$. Define a

gradation of \mathfrak{g} with respect to $\frac{1}{2}h$ as well,

$$\mathfrak{g}^r = \{y \in \mathfrak{g} : [\frac{1}{2}h, y] = ry\}, \quad r \in \frac{1}{2}\mathbb{Z}, \quad (2.6.1)$$

and set

$$\mathfrak{g}_t^r = \mathfrak{g}_t \cap \mathfrak{g}^r. \quad (2.6.2)$$

Then

$$\mathfrak{g} = \bigoplus_{t,r} \mathfrak{g}_t^r. \quad (2.6.3)$$

Set

$$\mathfrak{m}^e = \bigoplus_{t=r} \mathfrak{g}_t^r, \quad \mathfrak{n}^e = \bigoplus_{t < r} \mathfrak{g}_t^r, \quad \mathfrak{p}^e = \mathfrak{m}^e \oplus \mathfrak{n}^e. \quad (2.6.4)$$

Clearly, $\mathfrak{a} \subset \mathfrak{g}_0^0 \subset \mathfrak{m}^e$.

Definition. One says that χ is rigid for a Levi subalgebra \mathfrak{m} , if χ is congruent modulo $\mathfrak{z}(\mathfrak{m})$ to a middle element of a nilpotent orbit in \mathfrak{m} .

Whenever Q is a subgroup with Lie algebra \mathfrak{q} , we will write $Q_0 = Q \cap G_0$ and $\mathfrak{q}_t = \mathfrak{q} \cap \mathfrak{g}_t$.

We record the important properties of \mathfrak{p}^e .

Proposition ([L7]). Consider the subalgebra \mathfrak{p}^e defined by (2.6.4), and let P^e be the corresponding parabolic subgroup.

- (1) \mathfrak{p}^e depends only on e and not on the entire Lie triple $\{e, h, f\}$.
- (2) χ is rigid for \mathfrak{m}^e .
- (3) e is an element of the open M_0^e -orbit in \mathfrak{m}_1^e .
- (4) The P_0^e -orbit of e in \mathfrak{p}_1^e is open, dense in \mathfrak{p}^e .
- (5) $Z_{G_0}(e) \subset P^e$.
- (6) The inclusion $Z_{M_0^e}(e) \subset Z_{G_0}(e)$ induces an isomorphism of the component groups.

An immediate corollary of (4) and (5) in the proposition is a dimension formula for the orbits in $\text{Orb}_1(\chi)$.

Corollary (Lusztig). For an orbit $\mathcal{O}_e \in \text{Orb}_1(\chi)$,

$$\dim \mathcal{O}_e = \dim \mathfrak{p}_1^e - \dim \mathfrak{p}_0^e + \dim \mathfrak{g}_0, \quad (2.6.5)$$

where $\mathfrak{p}_i^e = \mathfrak{p}^e \cap \mathfrak{g}_i$, $i = 0, 1$.

2.7. Definition. A parabolic subgroup P with Lie algebra \mathfrak{p} is called good for χ if $\mathfrak{p} = \mathfrak{p}^e$ for some nilpotent $e \in \mathfrak{g}_1$ (notation as in (2.6.4)), and such that it satisfies (2) in proposition 2.6.

Let $\mathcal{P}(\chi)$ denote the set of good parabolic subgroups for χ . The parameterization of $\text{Orb}_1(\chi)$ is as follows.

Theorem ([L7]). The map $\mathcal{O}_e \mapsto P^e$ defined in section 2.6 induces a bijection between $\text{Orb}_1(\chi)$ and G_0 -conjugacy classes in $\mathcal{P}(\chi)$.

Proof. The definition of the inverse map is at follows. Let $P = MN$ be a good parabolic for χ . Then there exists s a middle element of a Lie triple in \mathfrak{m} , such that $\chi \equiv s \pmod{\mathfrak{z}(\mathfrak{m})}$. Moreover, the decomposition (2.6.4) must hold with respect to χ and s . Let $G_0^0 \subset G_0$ be the reductive subgroup whose Lie algebra is \mathfrak{g}_0^0 (notation as in section 2.6). Then G_0^0 acts on \mathfrak{g}_1^1 , and there is a unique open orbit of this action.

Let \mathcal{O} be the unique G_0 -orbit on \mathfrak{g}_1 containing it. The inverse map associates \mathcal{O} to P . \square

3. REDUCIBILITY POINTS

3.1. Let $\{e, h, f\}$ be a graded Lie triple for the orbit $\mathcal{O}_e \in \text{Orb}_1(\chi)$. Assume that $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is a standard parabolic subalgebra, $\mathfrak{b} \subset \mathfrak{p}$, such that $\{e, h, f\} \subset \mathfrak{m}$. Let $\bar{\mathfrak{p}} = \mathfrak{m} + \bar{\mathfrak{n}}$ be the opposite parabolics subalgebra. Let $\Pi_P \subset \Pi$ denote the simple roots defining P , and denote by Δ_M and Δ_N the roots in \mathfrak{m} , respectively \mathfrak{n} . We can write

$$\chi = \frac{1}{2}h + \underline{\nu}, \text{ with } \underline{\nu} \in \mathfrak{z}_{\mathfrak{g}}(e, h, f).$$

Lemma. *Let $\{e, h, f\}, \chi$ be as before, and assume that $\chi = \frac{1}{2}h + \underline{\nu}$ has $\underline{\nu}$ dominant with respect to Δ_N . Then:*

- (1) $\mathfrak{m}^e = \mathfrak{m} = \mathfrak{z}_{\mathfrak{g}}(\underline{\nu})$.
- (2) $\mathfrak{p}^e = \bar{\mathfrak{p}}$.

In particular, $\bar{\mathfrak{p}}$ is a good parabolic for χ .

Proof. The first assertion is obvious by the definitions. From (2.6.4) and the dominance conditions, we also see immediately that $\bar{\mathfrak{n}} = \mathfrak{n}^e$. \square

Let σ be the tempered module of \mathbb{H}_{M_s} (notation as in 1.3) parameterized by $\{e, h, f\}$. By the classification theorems of [L3] and [L4], we know that, in the correspondences of theorem 2.2, the standard module $X(P, \sigma, \nu)$, and the Langlands quotient $L(P, \sigma, \nu)$ are parameterized in $\text{Orb}_1(\chi)$ by the orbit $G_0 \cdot e$. Therefore, in theorem 2.7, they correspond to the parabolic subalgebra $\bar{\mathfrak{p}}$.

3.2. Now assume that $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is a *maximal* parabolic of \mathfrak{g} . Then $\Pi \setminus \Pi_P = \{\alpha\}$. Let $\tilde{\omega}$ denote the fundamental coweight for α .

As before, let σ be the tempered module attached to the map

$$\mathfrak{sl}(2) = \mathbb{C}\langle e, h, f \rangle \hookrightarrow \mathfrak{m}. \quad (3.2.1)$$

Then \mathfrak{n} is an $\mathfrak{sl}(2)$ -module, via the adjoint action of \mathfrak{m} . Let $k(\alpha)$ denote the multiplicity with which α appears in the highest root for Δ .¹

The coweight $\tilde{\omega}$ commutes with the $\mathfrak{sl}(2)$. Decompose \mathfrak{n} as $\mathfrak{n} = \bigoplus_{i=1}^{k(\alpha)} \mathfrak{n}_i$, where \mathfrak{n}_i is the i -eigenspace of $\tilde{\omega}$. Then decompose each \mathfrak{n}_i into simple $\mathfrak{sl}(2)$ -modules

$$\mathfrak{n}_i = \bigoplus_j (d_{ij}), \quad i = 1, \dots, k(\alpha), \quad (3.2.2)$$

where (d) is the simple $\mathfrak{sl}(2)$ -module of dimension d .

Theorem. *Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be a maximal parabolic, and σ be a generic tempered module parameterized by (3.2.1). Then the reducibility points $\nu > 0$ of the standard \mathbb{H}_0 -module $X(P, \sigma, \nu)$ are*

$$\nu \in \left\{ \frac{d_{ij} + 1}{2i} \right\}_{i,j}, \quad (3.2.3)$$

where the integers d_{ij} are defined in (3.2.2). Equivalently, these are the zeros of the rational function in ν ,

$$\prod_{\beta \in \Delta_N} \frac{1 - \langle \beta, \chi \rangle}{\langle \beta, \chi \rangle}, \quad (3.2.4)$$

¹If \mathfrak{g} is a classical simple algebra, this multiplicity is always 1 or 2.

where $\chi = \frac{1}{2}h + \nu\tilde{\omega}$ is the infinitesimal character of $X(P, \sigma, \nu)$.

Proof. Let $\mathcal{O}(\bar{\mathfrak{p}})$ be the orbit parameterizing $X(P, \sigma, \nu)$ by section 3.1. By section 2.5, $X(P, \sigma, \nu)$ is irreducible if and only if $\mathcal{O}(\bar{\mathfrak{p}}) = \mathcal{O}_{open}$.

Corollary 2.6 implies that $\dim \mathcal{O}(\bar{\mathfrak{p}}) = \dim \mathfrak{g}_0 - \dim(\mathfrak{g}_0 \cap \bar{\mathfrak{p}}) + \dim(\mathfrak{g}_1 \cap \bar{\mathfrak{p}})$. From this and the fact that $\dim \mathcal{O}_{open} = \dim \mathfrak{g}_1$, it follows, by equation (2.2.2), that $\mathcal{O}(\bar{\mathfrak{p}}) = \mathcal{O}_{open}$ if and only if

$$\#\{\beta \in \Delta_N : \langle \beta, \chi \rangle = 1\} = \#\{\beta \in \Delta_N : \langle \beta, \chi \rangle = 0\}. \quad (3.2.5)$$

Consider the rational function of ν , $\prod_{\beta \in R_N} \frac{1 - \langle \beta, \chi \rangle}{\langle \beta, \chi \rangle}$. Therefore, the reducibility points are given by the zeros of this function.

The explicit list of reducibility points follows from the fact that $\{\langle \beta, h \rangle : \beta \in R_N\} = \sqcup_{i,j} \{d_{ij} - 1, d_{ij} - 3, \dots, -d_{ij} + 1\}$, and so

$$\prod_{\beta \in \Delta_N} \frac{1 - \langle \chi, \beta \rangle}{\langle \chi, \beta \rangle} = \prod_{i,j} \frac{\frac{d_{ij}+1}{2i} - \nu}{\frac{d_{ij}-1}{2i} + \nu}. \quad (3.2.6)$$

□

We remark that in the proof of formula (3.2.4), one does not use the assumption that \mathfrak{p} be maximal parabolic. This formula holds as is for any parabolic \mathfrak{p} .

Example. The most interesting example of reducibility points for maximal parabolic induction is the case $\Pi_P = A_4 + A_2 + A_1$ in $\Pi = E_8$, with e the principal nilpotent in $A_4 + A_2 + A_1$ (which means that σ is the Steinberg representation). Then $k(\alpha) = 6$, $\dim \mathfrak{n} = 106$, and the $sl(2)$ decompositions (3.2.2) are

$$\begin{aligned} \mathfrak{n}_1 &= (8) + 2 \cdot (6) + 2 \cdot (4) + (2) & \mathfrak{n}_2 &= (9) + (7) + 2 \cdot (5) + (3) + (1) \\ \mathfrak{n}_3 &= (8) + (6) + (4) + (2) & \mathfrak{n}_4 &= (7) + (5) + (3) \\ \mathfrak{n}_5 &= (4) + (2) & \mathfrak{n}_6 &= (5). \end{aligned} \quad (3.2.7)$$

There are 11 reducibility points:

$$\left\{ \frac{3}{10}, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, 1, \frac{7}{6}, \frac{3}{2}, 2, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \right\}. \quad (3.2.8)$$

3.3. One also immediately obtains a partial result for non-generic data. Recall the notation and construction of section 2.1. In particular, if σ' is parameterized by (3.2.1), there exists a unique triple (S, \mathcal{C}, Ξ) such that σ' is a discrete series for the subalgebra $\mathbb{H}_{S, \Pi_{P/S}}$ in \mathbb{H}_S .

Proposition. *Let σ and σ' be tempered modules in the L -packet parameterized by (3.2.1), and assume that σ is generic. The standard \mathbb{H}_S -module $X(P/S, \sigma', \nu)$ is reducible for $\nu > 0$ only if the standard \mathbb{H}_0 -module $X(P, \sigma, \nu)$ is reducible.*

Proof. If $X(P/S, \sigma', \nu)$ is reducible, then the corresponding orbit is not the open orbit. But this means $X(P, \sigma, \nu)$ is reducible as well. □

Remark. This result gives necessary conditions for reducibility, but not sufficient. In fact, they are far from being sharp for non-generic inducing data as seen in the following example.

Example. Consider \mathbb{H}_0 of type C_{n+1} , and \mathfrak{p} of type C_n , and assume that n is a triangular number. Let the nilpotent element e correspond to the distinguished orbit $(2, 4, \dots, 2k)$ in $\mathfrak{sp}(2n)$, and χ be half the middle element of a Lie triple for e .

There are $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ discrete series in $\text{mod}_\chi \mathbb{H}_0(C_n)$. Let σ be the generic one. There exists a discrete series, call it σ' , characterized by the fact that $\sigma'|_{W(C_n)} = \mu_k$, where

$$\mu_k = \begin{cases} m^{2m+1} \times 0, & \text{if } k = 2m \\ 0 \times (m+1)^{2m+1}, & \text{if } k = 2m+1. \end{cases} \quad (3.3.1)$$

(The notation for $W(C_n)$ -representations, and the algorithms necessary for the computation are as in [L5].)

Theorem 3.2 implies that the reducibility points, $\nu > 0$, for $X(C_n, \sigma, \nu)$ are

$$\nu \in \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots, k + \frac{1}{2} \right\},$$

but one can show that the reducibility points of $X(C_n, \sigma', \nu)$ are just

$$\nu \in \left\{ \left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2}, k + \frac{1}{2} \right\}.$$

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